The Delta-Epsilon Way

Introduction to Limits/Closeness

The Concept - A number L is called the limit of a function $f(x)$ as the value of x approaches some constant **c** when the following condition is true: if for every positive number ε (*epsilon*), there exists an associated positive number δ (*delta*) such that if the distance between **c** and x is less than d, then the distance between the number L and *f(x)* is less than e. In other words,

$$|f(x) - \mathrm{L}| < \varepsilon \quad \text{whenever} \quad 0 < |x - \mathrm{c}| < \delta$$

(See diagram below.)

Example #1

The Problem:

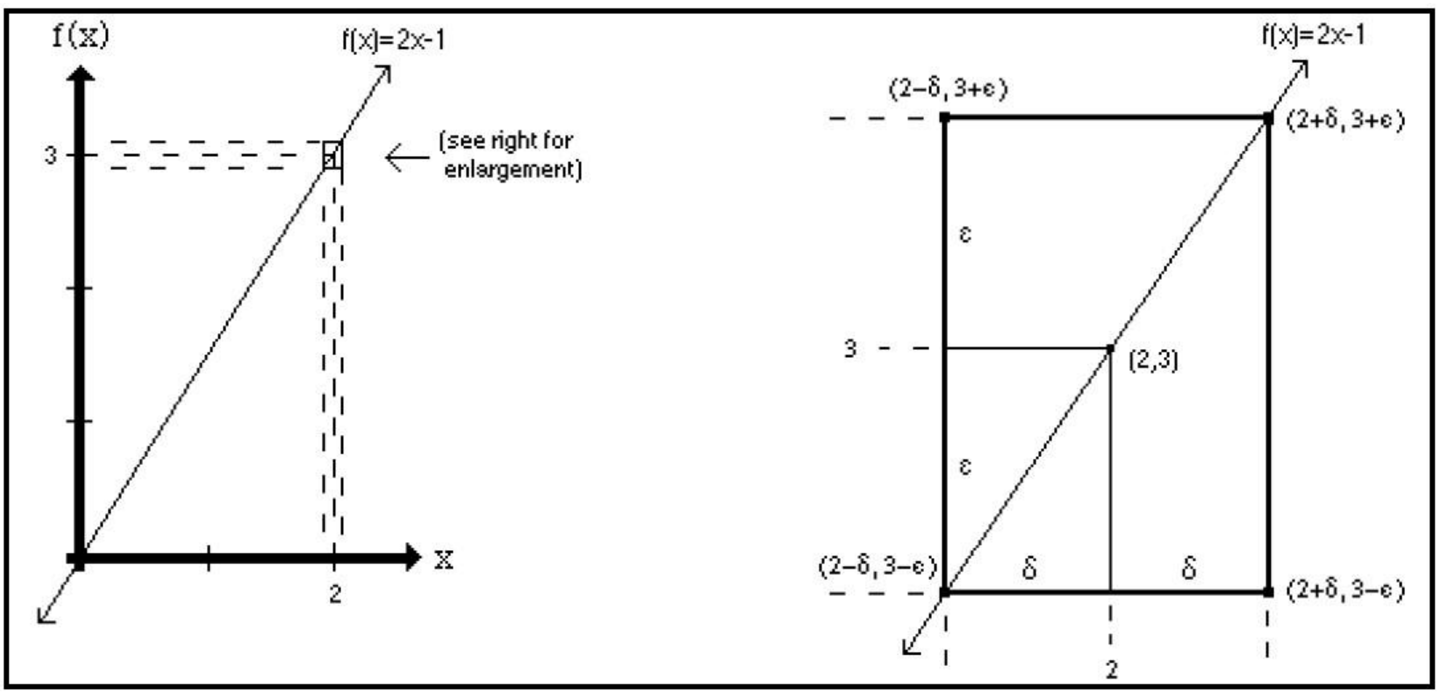
Verify the existence of the following limit.

$$\lim_{x \to 2} (2x - 1) = 3$$

The Procedure:

Given ε, find a δ given that $f(x) = 2x - 1$, $\mathrm{c} = 2$, and $\mathrm{L} = 3$. Use $|f(x) - \mathrm{L}| < \varepsilon$ (which means: $L - \varepsilon < f(x) < L + \varepsilon$, see diagram below) to find a δ. (Note: δ is usually written as an expression in terms of ε.)

Graphical Interpretation:



The Math:

Need to show that: for every ε there exists a δ such that

$|f(x) - 3| < \varepsilon$ (what is given) whenever $0 < |x - 2| < \delta$ (need to verify)

The Proof:

$|f(x) - \mathrm{L}| < \varepsilon$ *Hint #1: substitute for f(x) and L.*

$|(2x - 1) - 3| < \varepsilon$ *Hint #2: manipulate $|f(x) - L|$ (using algebra)*

$|2x - 4| < \varepsilon$ *so that it looks like $|x - c| < \delta$.*

$|2(x - 2)| < \varepsilon$

$2 \cdot |x - 2| < \varepsilon$

$|x - 2| < \frac{\varepsilon}{2}$ *Now let $\delta = \frac{\varepsilon}{2}$.*

$|x - 2| < \delta$ *Hint #3: We must finally obtain the form $|x - 2| < \delta$.*

Example #2

The Problem: Verify the existence of the following limit.

$$\lim_{x \to 2} \left(\frac{6x^2 + x - 2}{3x + 2} \right)$$

The Procedure: Always ***factor*** the numerator and denominator to see if any cancellations occur.

The Solution:

$$\lim_{x \to 2} \left(\frac{6x^2 + x - 2}{3x + 2} \right)$$

$$= \lim_{x \to 2} \left(\frac{(3x+2)(2x-1)}{3x+2} \right)$$ *Hint: cancel factors.*

$$= \lim_{x \to 2} (2x-1)$$

Example #3

The Problem:

Given that $f(x) = x^2 + x - 7$, verify that $\lim_{x \to 3} f(x) = 5$.

The Procedure:

$$\text{Given } |f(x) - 5| < \varepsilon \text{ find a } \delta \text{ such that } 0 < |x - 3| < \delta.$$

The Proof:

step #1: $|f(x) - L| < \varepsilon$ *Hint #1: substitute for f(x) and L.*

step #2: $|(x^2 + x - 7) - 5| < \varepsilon$

step #3: $|x^2 + x - 12| < \varepsilon$

step #4: $|x + 4| \cdot |x - 3| < \varepsilon$

Since $x \to 3$ in our problem, we can assume that x is close to 3. So, since x is in the vicinity of 3, *assume* that $|x - 3| \leq 1$ (i.e., the *distance* between x and 3 is less than or equal to 1). This is done for convenience and is sometimes called a *preliminary assumption*.

step #5: $|x - 3| \leq 1$

step #6: $-1 \leq x - 3 \leq 1$ *Hint #2: isolate x by adding 3 to all parts.*

step #7: $2 \leq x \leq 4$

So, 2 and 4 are the extreme values for x. Now, choose the extreme value that will make $|x + 4|$ the *largest*. This is done in order to make the quantity $\frac{\varepsilon}{|x+4|}$ the *smallest*. So, $|2 + 4| = 6$ or $|4 + 4| = 8$; we pick $x = 4$ since $8 > 6$.

step #8: $|x + 4| \cdot |x - 3| < \varepsilon$ *(from step #4)*

step #9: $|4 + 4| \cdot |x - 3| < \varepsilon$

step #10: $(8) \cdot |x - 3| < \varepsilon$

step #11: $|x - 3| < \frac{\varepsilon}{8}$

In step #5 it was assumed that $|x-3| \leq 1$, and in step #11 the result was $|x-3| < \frac{\varepsilon}{8}$. To ensure that $|x-3| < \delta$ implies that $|f(x)-5| < \varepsilon$, we need δ to be the *minimum* of 1 and $\frac{\varepsilon}{8}$, i.e., δ is *smaller* of 1 and $\frac{\varepsilon}{8}$. So, let $\boxed{\delta = \min(1, \frac{\varepsilon}{8})}$.